Eur. Phys. J. B **36**, 129–136 (2003)
DOI: 10.1140/epjb/e2003-00325-x

THE EUROPEAN PHYSICAL JOURNAL B

The complexity of the spherical p-spin spin glass model, revisited

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Received 24 July 2003
Published online 19 November 2003 –

Abstract. Some questions concerning the calculation of the number of "physical" (metastable) states or complexity of the spherical p-spin spin glass model are reviewed and examined further. Particular attention is focused on the general calculation procedure which is discussed step-by-step.

PACS. 75.10.Nr Spin-glass and other random models – 02.30.Mv Approximations and expansions

1 introduction

The analysis of the equilibrium and non-equilibrium properties in terms of the energy landscape originally pushed forward for the structural glass transition [1], has risen in the recent years a new interest on the topological properties of the energy or free-energy landscapes of disordered and complex systems. In this approach an important theoretical tool is the logarithm of the number of metastable states, called *complexity* or *configurational entropy*, identified as basins or valleys on the landscape.

Solvable models, such as mean-field models, have always played an important role in the theoretical study of physical problems. In this context classical calculations [2,3] for the complexity of the Sherrington-Kirkpatrick [4] (SK) and other disordered spin models have been reconsidered, extended and in some cases criticized [5–9].

Motivated by these criticisms, in this paper some questions concerning the calculation of the complexity of disordered spin systems are reviewed and examined further in a solvable model, the spherical p-spin spin glass (pSP-SG) model introduced by Crisanti and Sommers [10]. The aim of the paper is not the calculation of the complexity for the spherical pSP-SG model, which has been computed in references [11–13], but the procedure of calculation itself to clarify known results which could be useful for the understanding of the complexity of other systems. The spherical pSP-SG model is only used to enlighten subtle points of the procedure. We have tried to make the paper almost self-contained so that it can also be used by readers interested into the problem but not too familiar with all reported results.

The general approach to the calculation of complexity in mean-field spin glass model is discussed in Section 2. The procedure is illustrated in Section 4 using the spherical pSP-SG model introduced in Section 3. The correctness the procedure is discussed in Section 5. Finally Section 6 contains some conclusions and discussion.

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2 How to compute the complexity

The metastable states in mean-field spin models are more easily studied using the Thouless-Anderson-Palmer (TAP) method [14], which introduces a mean-field free energy functional $F_{\rm TAP}(\boldsymbol{m})$ of the local magnetizations $\boldsymbol{m} = (m_1, m_2, \ldots, m_N)$, where N is the number of spins. At any temperature T the metastable states of the system are identified with the local minima of $F_{\rm TAP}(\boldsymbol{m})$, i.e., with the solutions of

$$\partial_{m_i} F_{\rm TAP}(\boldsymbol{m}) = 0, \qquad i = 1, \ldots, N \tag{1}$$

with the additional requirement that *all* eigenvalues of the matrix $\partial_{m_i}\partial_{m_j} F_{\rm TAP}(\boldsymbol{m})$ evaluated on the solution are *positive*. At the mean-field level different local minima are separated by infinite barriers, therefore the system cannot escape from a local minimum in a finite time and hence the minimum (and its basin of attraction) is a metastable state of infinite life-time. However, despite this simple intuitive picture, not all minima of $F_{\rm TAP}(\boldsymbol{m})$ can be associated with *physical* metastable states but only those for which (Plefka's criterion) [14–16]

$$x_{\rm P} = 1 - \frac{c(q)}{N} \sum_{i=1}^{N} \left(1 - m_i^2\right)^2 \geq 0 \tag{2}$$

where $\beta = 1/T$ and $c(q)$ is a function of $q = (1/N)\sum_i m_i^2$ whose form depends on the interactions. For example for

the SK model $c(q) = \beta^2$ [14–16], while for p-spin interaction models $c(q) = (\beta^2 p/2)(p-1)q^{p-2}$ [17]. This condition guarantees that TAP solutions display a physical local susceptibility equal to $\beta(1-q)$. In conclusion within this approach the calculation of the number of metastable states is reduced to that of counting the number of solutions of (1) which are minima and satisfies the Plefka's criterion (2) ("physical" minima). If it were found that physical minima must satisfy additional constraints, those must also be included.

Different physical minima may have different free-energy density, thus to have a better description of metastable states one can group together all minima with the same free-energy density and introduce the function $\rho(f)$ which gives the number of metastable states with $F_{\rm TAP}(\boldsymbol{m}) = Nf$. The configurational entropy is then defined as:

$$\Sigma(f) = \frac{1}{N} \ln \rho(f). \tag{3}$$

We are eventually interested into the large N limit, thus $\Sigma(f)$ is different from zero only if the number of physical minima with free energy density f is exponentially large with N.

If we label the $\mathcal{N}_{\rm sol}$ solutions of the TAP equation (1) with the subscript α ($\alpha = 1, \ldots, \mathcal{N}_{\rm sol}$) by definition $\rho(f)$ is given by

$$\rho(f) \stackrel{def}{=} \sum_{\alpha=1}^{\mathcal{N}_{\rm sol}} \prod_{i=1}^{N} \left[\int dm_i \, \theta(\lambda_i^\alpha) \, \delta(m_i - m_i^\alpha) \right] \times \theta(x_{\rm p}^\alpha) \, \delta\left[F_{\rm TAP}(\boldsymbol{m}) - Nf\right] \tag{4}$$

where λ_i^α ($i = 1, \ldots, N$) are the eigenvalues of the Hessian matrix for the αth solution:

$$\chi_{ij}^\alpha = \partial_{m_i} \partial_{m_j} F_{\rm TAP}(\boldsymbol{m})\big|_{\boldsymbol{m}=\boldsymbol{m}^\alpha} \tag{5}$$

and $\theta(x)$ is the Heaviside theta-function. As it stands (4) is difficult to handle, however using the properties of delta-function it can be transformed into the more manageable form:

$$\rho(f) = \prod_{i=1}^{N} \left[\int dm_i \, \theta(\lambda_i) \, \delta\left[\partial_{m_i} F_{\rm TAP}(\boldsymbol{m})\right] \right] \times \det(\underline{\chi}(\boldsymbol{m})) \, \theta(x_{\rm P}) \, \delta\left[F_{\rm TAP}(\boldsymbol{m}) - Nf\right] \tag{6}$$

where $\underline{\chi}(\boldsymbol{m})$, the Hessian matrix (5) evaluated for a generic $\boldsymbol{m}$, is the Jacobian of the transformation and λ_i are its eigenvalues. The theta-functions ensure that the determinant of χ is always *positive* and we have neglected the absolute value of the Jacobian.

In addition to (4) we consider the definition without the theta-functions, which we denote by $\rho_{\rm tot}(f)$, which counts the total number of TAP solutions. The effect of the theta-functions is to eliminate all solutions with at least one negative eigenvalue, therefore the meaning of $\rho_{\rm tot}(f)$ is not exactly the same as that of $\rho(f)$ since all solutions are now counted. There is just one case in which the two formulations, at least in the limit of our interest, $N \gg 1$, are indeed equivalent: if for large N the two integral – with and without theta-functions – are dominated by the *same set* of solutions, an assumption that must be verified in each case (and for each value of f) separately.

Keeping the sum over all solutions is not, however, completely free of difficulties: since all solutions are counted the determinant of the Jacobian can be negative and the absolute value must be retained making the subsequent calculation more problematic. To overcome this difficulties the absolute value is simply dropped leading to expression:

$$\tilde{\rho}(f) = \prod_{i=1}^{N} \left[\int dm_i \, \delta\left[\partial_{m_i} F_{\rm TAP}(\boldsymbol{m})\right] \right] \times \det(\underline{\chi}(\boldsymbol{m})) \, \delta\left[F_{\rm TAP}(\boldsymbol{m}) - Nf\right], \tag{7}$$

and arguments are given to justify under which circumstances this reproduces the correct result with the absolute value.

We have to compare $\tilde{\rho}(f)$ given by (7) with $\rho(f)$. The question is when $\tilde{\rho}(f)$ yields the same result as $\rho(f)$. The main difference between (6) and (7) is the support of the integrals, larger for the latter, hence the two expressions are equivalent if the integrals are dominated by the *same support*. Thus, to extract from $\tilde{\rho}(f)$ the correct result for $\rho(f)$ we should be able to isolate the contributions from the common support. For a generic value of N this could be quite a hard problem. However, in the limit of large N where the integrals are evaluated by saddle point methods, a simple rule can be applied.

In this case $\rho(f)$ can be evaluated simply considering only the stationary points for which all eigenvalues of the Hessian are positive and the Plefka's criterion is satisfied, disregarding all others. We stress that such constraint is not contained into $\tilde{\rho}(f)$, so that the functional alone cannot give the desired result.

In the next sections we shall illustrate this procedure (re)computing the complexity for the spherical pSP-SG model without external field [11] using both expressions (7) and (6).

3 TAP Equations for the spherical pSP-SG model

The spherical pSP-SG model consists of N continuous spins σ_i interacting via p-body interactions [10]:

$$H(\sigma) = \frac{r}{2} \sum_{i=1}^{N} \sigma_i^2 - \sum_{1 \le i_1 < \cdots < i_p \le N} J_{i_1,\ldots,i_p} \, \sigma_{i_1} \cdots \sigma_{i_p}. \tag{8}$$

The couplings are quenched independent Gaussian variables with zero mean and average $\langle (J_{i_1,\ldots,i_p})^2 \rangle = p!/(2N^{p-1})$. The scaling with N ensures a well defined thermodynamic limit. Here and in the following $\langle (\cdots) \rangle$ denotes disorder average. The parameter r is a Lagrange multiplier to impose the global constraint $\sum_{i=1}^N \sigma_i^2 = N$ on the spins amplitude.

The study of both the static and dynamical properties shows that in the thermodynamic limit the model presents a (static) transition at a temperature $T_{\rm s}$, between a high temperature replica symmetric phase and a low temperature phase with one step of replica symmetry breaking [10]. Despite its simplicity, the spherical pSP-SG model for $p > 2$ has an exponentially large number of locally stable states which dominate the dynamical behavior above $T_{\rm s}$. As a consequence, two-time correlation functions acquire a time persistent part at a temperature $T_{\rm d} > T_{\rm s}$ which marks the dynamical transition [18]. The static transition can be seen as the point where the lowest accessible (metastable) states dominate. The dynamical transition, on the contrary, takes place at the point where the behavior is ruled by higher, highly degenerate, metastable states.

The TAP functional has been derived in references [11,19]:

$$\begin{aligned}\beta F_{\rm TAP}(\boldsymbol{m}) = &-\frac{\beta}{p!}\sum_{i_1,\ldots,i_p} J_{i_1,\ldots,i_p}\, m_{i_1}\cdots m_{i_p}\\ &-\frac{N}{2}\ln(1-q)\\ &-\frac{N\beta^2}{4}\left[1+(p-1)q^p-pq^{p-1}\right] \qquad (9)\end{aligned}$$

where $Nq = \sum m_i^2$, and taking the derivatives with respect to m_i one obtains the TAP equations.

The structure of the solutions is better understood performing the change of variable $m_i = q^{1/2}\widehat{m}_i$ ($\sum \widehat{m}_i^2 = N$) which leads to TAP functional density:

$$\begin{aligned}f_{\rm TAP}(q,E) = q^{p/2}\,E &-\frac{T}{2}\ln(1-q)\\ &-\frac{\beta}{4}\left[1+(p-1)q^p-pq^{p-1}\right] \qquad (10)\end{aligned}$$

where $E = -(1/Np!)\sum J_{i_1,\ldots,i_p}\widehat{m}_{i_1}\cdots\widehat{m}_{i_p}$ is the $T = 0$ energy density. In general, E is a random variable which depends on both the realization of couplings and on the orientation of the vector $\boldsymbol{m}$. However all cases with the same value of E will also have the same free energy, thus we can consider E as given and study the solutions as a function of E. The TAP equations then reduce to $\partial_q f_{\rm TAP}(q,E) = 0$ which can be written:

$$(1-q)\,q^{p/2-1} = zT \qquad (11)$$

where

$$z = \frac{1}{p-1}\left[-E \pm \sqrt{E^2 - E_{\rm c}^2}\right], \qquad (12)$$

$$E_{\rm c} = -\sqrt{2\,(p-1)/p}. \qquad (13)$$

It is easy to understand that for any positive z and temperature T below

$$T_{\rm a} = (1-q_{\rm a})\,q_{\rm a}^{p/2-1}\,z^{-1} \qquad (14)$$

where $q_{\rm a} = (p-2)/p$, there are two solutions of the TAP equation (11), one larger and one smaller than $q_{\rm a}$. Anyway, possible candidates for physical solutions are only those which are local minima of $f_{\rm TAP}(q,E)$.

By using the TAP equation (11) the second derivative of $f_{\rm TAP}(q,E)$ with respect to q evaluated on the solutions can be expressed as

$$\begin{aligned}\partial_q^2 f_{\rm TAP}(q,E) =&\frac{p}{4\beta q}\left[q-\frac{p-2}{p}\right]\\ &\times\left[\frac{1}{(1-q)^2}-\mu(p-1)q^{p-2}\right]\\ =&\frac{p}{4\beta q}\left[q-\frac{p-2}{p}\right]\frac{z_{\rm c}^2-z^2}{z_{\rm c}^2} \qquad (15)\end{aligned}$$

where $z_{\rm c} = \sqrt{2/p(p-1)}$ and $\mu = \beta^2 p/2$. The requirement of positiveness of the second derivative thus selects the solutions

$$q < \frac{p-2}{p} \quad \text{for } z > z_{\rm c} \qquad (16)$$

$$q > \frac{p-2}{p} \quad \text{for } z < z_{\rm c}. \qquad (17)$$

By comparing the two expressions in (15) we see that the condition $z < z_{\rm c}$ is equivalent to

$$x_{\rm P} = 1-\mu(p-1)q^{p-2}\,(1-q)^2 > 0 \qquad (18)$$

which also follows from the stability requirement of the replica saddle point [10] and of the dynamics [18]. This is the Plefka's criterion (2) for the physical relevance of TAP solutions [15] for the spherical pSP-SG model. Indeed it can be easily seen that the condition (16), for which $x_{\rm P} < 0$, leads to an unphysical q decreasing with temperature.

4 Complexity of the spherical pSP-SG model: standard calculation

In this section we report the main steps of the calculation of $\tilde{\rho}(f)$ for the spherical pSP-SG model. Details can be found in the literature, see e.g. references [2,3] and for the specific case of the p-spin spin glass model references [11,13,20].

The starting point is (cf. (7))

$$\begin{aligned}\tilde{\rho}(f) = N^2\int_0^1 dq\prod_{i=1}^{N}\left[\int_{-\infty}^{+\infty} dm_i\,\delta(G_i)\right]\det\underline{A}\\ \times\,\delta\left(Nq-\sum_i m_i^2\right)\delta\left[F_{\rm TAP}(\boldsymbol{m})-Nf\right] \qquad (19)\end{aligned}$$

with

$$\begin{aligned}G_i &= \partial_{m_i}\beta f_{\rm TAP}(\boldsymbol{m})\\ &= a(q)\,m_i - \frac{\beta}{(p-1)!}\sum_{\boldsymbol{j}} J_{i,\boldsymbol{j}}\,m^{p-1} \qquad (20)\end{aligned}$$

and

$$\begin{aligned} A_{ij} =&\partial_{m_j} G_i \\ =&a(q)\,\delta_{ij} - \frac{\beta}{(p-2)!}\sum_{\boldsymbol{k}} J_{ij,\boldsymbol{k}}\, m^{p-2} \\ &+ \frac{2}{N}\,a'(q) m_i\, m_j \end{aligned} \tag{21}$$

where

$$a(q) = \frac{1}{1-q} + \mu(p-1)(1-q)q^{p-2} \tag{22}$$

$a'(q) = da(q)/dq$ and we have used the short-hand notation:

$$\sum_{\boldsymbol{j}} J_{i,\boldsymbol{j}}\, m^{p-1} \stackrel{def}{=} \sum_{k_1,\ldots,k_{p-1}} J_{i,k_1,\ldots,k_{p-1}}\, m_{k_1}\cdots m_{k_{p-1}} \tag{23}$$

and similarly in (21). The last term of A is of order $O(1/N)$, and can be neglected for $N \to \infty$ (see also below).

The structure of the minima is given by the couplings, therefore $\tilde{\rho}(f)$ (and so $\rho(f)$) may change from sample to sample. Thus, in principle, to have a well defined complexity we should introduce replicas to compute $\langle \ln \tilde{\rho}(f)\rangle$ [21]. However it can be shown [11,12] that for this model, in absence of a magnetic field, the annealed average $\ln\langle\tilde{\rho}(f)\rangle$ is exact, so we can just average (19) over the disorder.

To perform the average over the couplings it is convenient to use the integral representation of the delta-function to exponentiate its argument. This introduces additional parameters which are usually denoted by $\hat{f}$, $\hat{q}$ and $\hat{m}_i$ [22] conjugated to f, q and G_i and the additional variable Δ coming from Hubbard-Stratonovich transformation. The calculation can be further simplified by substituting $\sum J_{i_1,\ldots,i_p}\, m_{i_1}\cdots m_{i_p}$ from equation (20) in $F_{\rm TAP}(\boldsymbol{m})$ (Eq. (9)) and noticing that the error involved in disorder-averaging the determinant of $\underline{A}$ separately accounts for changing $\underline{A}$ of terms of order $O(1/N)$ and hence negligible as $N \to \infty$ [2,20].

Performing the averages over the couplings results in

$$\langle\tilde{\rho}(f)\rangle = c\int_{-\infty}^{+\infty} d\hat{f}\int_0^1 dq\int_{-\infty}^{+\infty} d\hat{q}\int_{-\infty}^{+\infty} d\Delta\, e^{N\,\Sigma} \tag{24}$$

where c is a constant and

$$\begin{aligned} \Sigma = i\beta\hat{f}\left[f - f(q)\right] + i\hat{q}q - \Delta(1-q) \\ - \frac{1}{\lambda}\Delta^2 + \ln I + G_{x_{\rm p}}(q) \end{aligned} \tag{25}$$

with $f(q)$ the TAP density functional $f_{\rm TAP}(\boldsymbol{m})$ evaluated on the solution of the TAP equation (20):

$$\begin{aligned} f(q) = -\frac{\beta}{4}(1-q^p) - \frac{\beta}{4}(p-2)(1-q)q^{p-1} \\ - \frac{qT}{p\,(1-q)} - \frac{T}{2}\ln(1-q), \end{aligned} \tag{26}$$

$$\begin{aligned} I = \int_{-\infty}^{+\infty} \frac{dm\,d\hat{m}}{2\pi}\,\exp\Big\{\frac{\mu q^{p-1}}{2}(i\hat{m})^2 \\ + i\hat{m}\left(\frac{1}{1-q} - \Delta\right)m - i\hat{q}m^2\Big\}. \end{aligned} \tag{27}$$

and $\lambda = 2\mu(p-1)q^{p-2}$.

The function $G_{x_{\rm P}}(q)$ comes from the average of the determinant of $\underline{A}$ which can be computed either using Grassmann variables [3] or introducing replicas [2,20]. The form depends on the sign of $x_{\rm P}$ (Eq. (18)) [23]:

$$G_{x_{\rm P}}(q) = -\ln(1-q), \qquad \text{for } x_{\rm P} > 0 \tag{28}$$

this is $B = 0$ solution always adopted in standard calculations e.g. in [2,5,20] and

$$\begin{aligned} G_{x_{\rm P}} = \frac{1}{\lambda\,(1-q)^2}\left[1 - \frac{\lambda^2}{4}\,(1-q)^4\right] \\ + \ln\frac{\lambda}{2}(1-q), \qquad \text{for } x_{\rm P} < 0. \end{aligned} \tag{29}$$

The two expressions coincide for $x_{\rm P} = 0$, i.e. for $\lambda/2 = 1/(1-q)^2$. Details of the calculation can be found in Appendix A.

Integration over $\hat{m}, m, \hat{q}, \Delta$ can be done by the saddle point method [24], which turns out to be exact for the integrals over $\hat{m}, m, \Delta$ being Gaussian, while the integral over $\hat{f}$ can be easily performed giving a delta-function. The final results is then

$$\begin{aligned} \langle\tilde{\rho}(f)\rangle &\sim c'\int_0^1 dq\,\delta\big[f(q) - f\big]\,e^{N\,\Sigma(q)} \\ &\sim e^{N\,\Sigma(q^*)}, \qquad N \to \infty \end{aligned} \tag{30}$$

where

$$\begin{aligned} \Sigma(q) = G_{x_{\rm p}}(q) + \frac{1}{2} + \frac{1}{2}\ln q - \frac{1}{2}\ln(\mu q^{p-1}) \\ + \frac{p-1}{2\mu p q^{p-2}}\left[\frac{1}{1-q} - \mu(1-q)q^{p-2}\right]^2 - \frac{1}{2\mu(1-q)^2 q^{p-2}}. \end{aligned} \tag{31}$$

and $q^* = q^*(f)$ is the solution of

$$f(q) = f \tag{32}$$

which gives the largest value of $\Sigma(q)$ [25].

The simplest way of studying the solution is using q as a free parameter to scan all values of f. This is what is done, for example, in reference [11] where the result (31–32) was first derived.

The solutions of equation (32) can be found using the results of Section 3. The free energy $f(q)$ as function of q for all stationary points of $f_{\rm TAP}(q, E)$ is shown in Figure 1 for $p = 4$ and temperature T between the static transition temperature $T_{\rm s}$ and the dynamical transition temperature $T_{\rm d}$. Other values of p or T in this range lead to a qualitatively same picture. The corresponding $\Sigma(q^*)$ as function of f is shown in Figure 2.

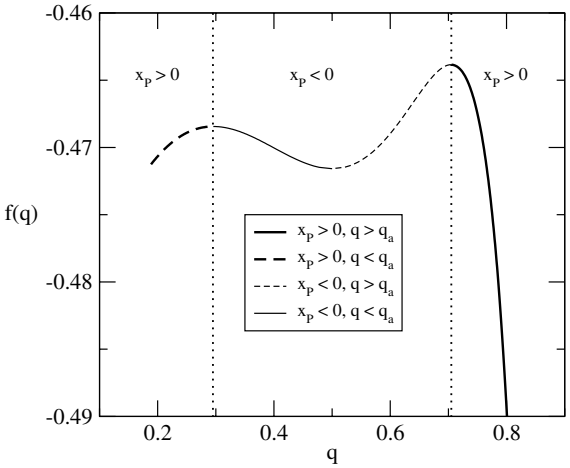

Fig. 1. Free energy density $f(q)$ as function of q for $p = 4$ and temperature $T = 0.51$ between the static transition temperature $T_{\rm s} = 0.5030...$ and the dynamical transition temperature $T_{\rm D} = 0.5443...$ Thicker lines correspond to solutions for which the Plefka's criterion is satisfied, while full lines correspond to solution for which (15) is positive, i.e., to local minima of $f_{\rm TAP}(q, E)$. Only the full thick line (last sector to the right) is relevant for the calculation of the number of metastable states, i.e., states which are both minima and have $x_{\rm p} > 0$.

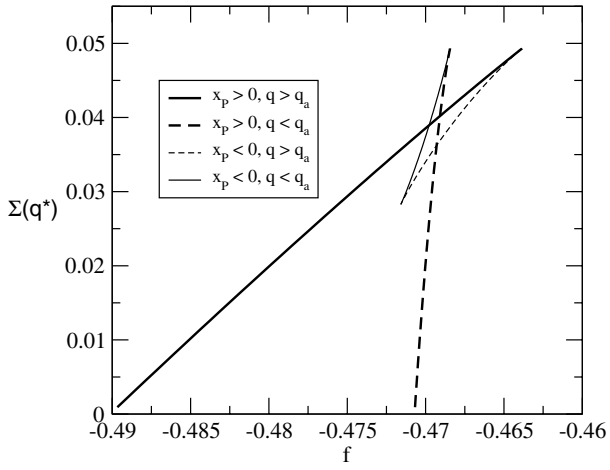

Fig. 2. $\Sigma(q^*)$ as as function of f for $p = 4$ and temperature $T = 0.51$ between the static transition temperature $T_{\rm s} = 0.5030...$ and the dynamical transition temperature $T_{\rm D} = 0.5443...$ Thicker lines correspond to solutions for which the Plefka's criterion is satisfied, while full lines correspond to solution for which (15) is positive, i.e., to local minima of $f_{\rm TAP}(q, E)$. Only the full thick line is relevant for the calculation of the number of metastable states, i.e., states which are both minima and have $x_{\rm p} > 0$. In the figure only values of f for which $\Sigma > 0$ are reported.

Strictly speaking to evaluate $\langle\tilde{\rho}(f)\rangle$ we should take for each value of f the largest value of Σ, and compute the sign of the neglected coefficient in (30). However, one is actually interested into the number of metastable states, so in all calculations done so far all solutions with $x_{\rm p} < 0$ are cut out, the "famous" $B = 0$ solution. Even if not explicitly stated, this is in the spirit of the procedure described in Section 2. We stress, however, that if the procedure is the same the motivations are not. Indeed the $B \neq 0$ solution can also describe minima of the TAP functional but such configurations violate the Plefka criterion ($x_{\rm p} < 0$), thus leading to a non-physical linear susceptibility.

If the $x_{\rm p} < 0$ solutions are disregarded, we are left with the curves shown in Figure 3 corresponding to the solution of the TAP equations with $q < (p-2)/p$ (dashed line) and $q > (p-2)/p$ (full line). Again, if no other information is added, for each f the largest value must be selected to evaluate $\Sigma(f)$. This means that there is a region of free energies where $\Sigma(f)$ is dominated by solutions with $q < (p-2)/p$. But these are not local minima of $f_{\rm TAP}$, see (15), and hence for these free energies $\Sigma(f)$ does not give the desired result. This clearly shows that the condition $x_{\rm p} > 0$ *alone* does not guarantees that only physical states are counted. To find the correct answer additional information on the solutions must be added.

For the spherical pSP-SG this information is easily obtained. Indeed, the analysis of the TAP solutions in Section 3, shows that only solutions with $x_{\rm p} > 0$ *and* $q > (p-2)/p$ do correspond to metastable states, so only the full line in Figure 3 must be considered. This leads to the result first derived in reference [11].

To conclude this section we note that for the pSP-SG model, using the TAP equation (11) $\Sigma(f)$, can be

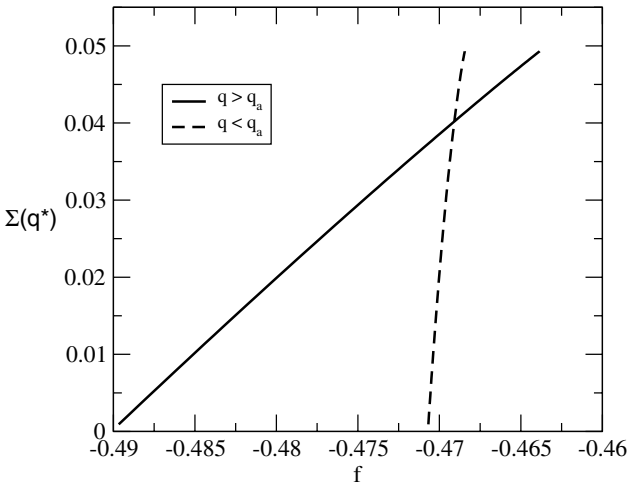

Fig. 3. Same as Figure 2 where only solutions satisfying the Plefka's criterion are displayed. The full line is the result found in reference [11].

rewritten as a function of z only:

$$\Sigma^{+}(z) = \frac{1}{2}\left[\frac{2-p}{p} - \ln\frac{pz^2}{2} + \frac{p-1}{p}z^2 - \frac{2}{p^2z^2}\right] \quad (33)$$

for $x_{\rm P} > 0$ [11], and

$$\Sigma^{-}(z) = \Sigma^{+}(z) + \ln\left[\frac{p(p-1)}{2}z^2\right] + \frac{1}{p(p-1)z^2}\left[1 - \frac{p^2(p-1)^2z^4}{4}\right] \quad (34)$$

for $x_{\rm P} < 0$. As a consequence the number of solutions in each sector $x_{\rm p} > 0$ ans $x_{\rm p} < 0$ is conserved under temperature changes.

5 Complexity of the spherical pSP-SG model: Hessian eigenvalues

In the previous section we have revised step-by-step the standard calculation of the complexity for the spherical pSP-SG model showing which additional information, not included into the definition of $\tilde{\rho}(f)$, must be added to yield the correct answer. In this section we show that the additional information is exactly the theta-functions needed to transform $\tilde{\rho}(f)$ into $\rho(f)$, see equations (6) and (7).

To prove the equivalence we must compute the eigenvalues λ_i of the Hessian matrix (5) that for the spherical pSP-SG model is given by (21).

The eigenvalues are solutions of the equations

$$\begin{aligned}\sum_j A_{ij}\,\xi_j =& a(q)\,\xi_i + 2\,q\,a'(q)\,\hat{m}_i\,\frac{1}{N}\,\sum_j \hat{m}_j\,\xi_j \\ &- \frac{\beta\,q^{(p-2)/2}}{(p-2)!}\sum_{j,\boldsymbol{k}} J_{ij,\boldsymbol{k}}\,\hat{m}^{p-2}\,\xi_j \\ =& \lambda\,\xi_i. \end{aligned} \tag{35}$$

There are two classes of eigenvectors ξ_i: longitudinal and transversal.

5.1 Longitudinal eigenvector

The longitudinal eigenvector is given by:

$$\xi_i \propto \hat{m}_i, \qquad \forall i \tag{36}$$

and hence satisfies the equation:

$$[a(q) + 2\,q\,a'(q)]\;\hat{m}_i \;-\; \frac{\beta\,q^{(p-2)/2}}{(p-2)!}\sum_{\boldsymbol{j}} J_{i,\boldsymbol{j}}\,\hat{m}^{p-1} = \lambda_{\rm L}\,\hat{m}_i. \tag{37}$$

Since the Hessian must be evaluated on the solution of the TAP equation (1), we can use (20) to write

$$\beta\,q^{(p-2)/2}\sum_{\boldsymbol{j}} J_{i,\boldsymbol{j}}\,\hat{m}^{p-1} = a(q)\,(p-1)!\,\hat{m}_i \tag{38}$$

which inserted into (37) leads to

$$\begin{aligned}\lambda_{\rm L} &= 2\,q\,a'(q) - (p-2)\,a(q) \\ &= p\left[q - \frac{p-2}{p}\right]\left[\frac{1}{(1-q)^2} - \mu(p-1)q^{p-2}\right]. \end{aligned} \tag{39}$$

The longitudinal eigenvalue is therefore, apart from positive multiplicative coefficients, equal to $\partial_q^2 f_{\rm TAP}(q,E)$ evaluated in Section 3 (Eq. (15)). The different coefficients come from the derivative being taken with respect to q or to m_i. The longitudinal eigenvalue has degeneracy 1.

Note that the term of $O(1/N)$ in (21) yields a contribution of $O(1)$ for longitudinal eigenvectors and cannot be neglected [15].

5.2 Transversal eigenvectors

Transversal eigenvectors satisfy the orthogonality conditions:

$$\sum_i \xi_i\,\hat{m}_i = 0 \tag{40}$$

and hence span a space of dimension $N-1$. The eigenvalues equation for transversal eigenvectors can be written as

$$a(q)\,\xi_i - \sum_j \widetilde{J}_{ij}\,\xi_j = \lambda_{\rm T}\,\xi_i \tag{41}$$

$$\widetilde{J}_{ij} = \beta\,q^{(p-2)/p}\sum_{k_1<\cdots<k_{p-2}} J_{ij,k_1,\ldots,k_{p-2}}\,\hat{m}_{k_1}\cdots\hat{m}_{k_{p-2}}. \tag{42}$$

For large values of N $\widetilde{J}_{ij}$ is a symmetric random matrix whose elements are independent Gaussian variables with zero average and variance:

$$\langle(\widetilde{J}_{ij})^2\rangle = \frac{\mu(p-1)q^{p-2}}{N}. \tag{43}$$

Therefore for $N\to\infty$ the spectrum of $\widetilde{J}_{ij}$ is given by the Wigner's semicircular law [13,26]:

$$\begin{aligned}\rho(\lambda_{\rm T}) = &\frac{1}{2\pi\mu(p-1)q^{p-2}} \\ &\times\sqrt{4\mu(p-1)q^{p-2} - [\lambda_{\rm T} - a(q)]^2}. \end{aligned} \tag{44}$$

This gives a spectrum at the leading order in N, displaying a non negative support. Since it can be shown that the tails of this distribution go to zero exponentially with N [26] we can safely exclude negative eigenvalues. The thermodynamic limit transversal fluctuations are, thus, always stable, regardless of the sign of $x_{\rm P}$ and the whole stability depends on the longitudinal eigenvalue. Note, however, that the $N-1$ transversal eigenvalues dominate the calculation of

$$\det \underline{A} = \exp\left(\mathrm{Tr}\ln \underline{A}\right) \qquad \text{for } N\to\infty \tag{45}$$

and any information from the longitudinal eigenvalue is washed out when computing $\tilde{\rho}(f)$.

In conclusion we see that the procedure described in Section 2 of selecting the saddle point solutions of $\tilde{\rho}(f)$ according to their physical relevance obtained from the (independent) analysis of the TAP equations produces the correct result for the complexity.

6 Conclusion and discussion

The study of the complex behavior of glassy systems in terms of the topological properties of the energy or free-energy surfaces has recently put new interest into the calculation of the number of metastable states, also called complexity or configurational entropy, in mean-field spin glass models. In this context classical calculations done

for the SK and other disordered spin models have been reconsidered, extended and also criticized [5–8].

Motivated by these controversies in this paper we have reviewed and examined further some questions concerning the calculation of the complexity of disordered systems. Particular care has been taken to distinguish between what we would like to compute, $\rho(f)$, and what we are able to compute, $\tilde{\rho}(f)$. We have also discussed how information on $\rho(f)$ can be extracted, at least in the thermodynamic limit.

The general approach has been illustrated using the spherical pSP-SG model, showing the correctness of the reduction procedure. As by-product we have explicitly shown that the Plefka criterion separates all solutions of the TAP equations into two classes, both containing local minima as well as saddles. However, only local minima which satisfy the Plefka criterion do represent *physical states*. The Plefka's criterion is indeed a *necessary*, but not sufficient, condition for physical states, and hence cannot be used alone for the reduction procedure but the requirement of local stability must be added.

In reference [5] the classical calculation of Bray and Moore [2] and DeDominicis and Young [3] for the SK model have been critically revised. The main criticism steams from the observation that $\tilde{\rho}(f)$ (identified with $\rho(f)$ in those papers) can be written as saddle point calculation over a functional which posses a supersymmetry between commuting and anti-commuting variables used to express the Jacobian in (7) [27]. The classical solution breaks this symmetry and in reference [5] a different, supersymmetric solution was proposed.

In a separate paper [6] we have performed a careful analysis of both the classical and the new supersymmetric solutions. The outcome is that both solutions have some limitations. For example, the neglected prefactor could be exponentially small in N for the classical solution changing its prediction [6,28]. On the other hand the supersymmetric solution has a negative $x_{\rm p}$, while it is positive for the classical solution. This may suggest the correctness of the classical solution over the supersymmetric one. However, as we have explicitly shown here for the spherical pSP-SG model, this condition is not a sufficient condition. In order to prove the correctness of the solution one should prove that it corresponds to a physical (stable) state. A rather difficult problem already at the annealed (replica symmetric) level used in these calculations. A step in this direction has been recently done in the paper by Aspelmeier, Bray and Moore [29] where the Hessian of the fluctuations for the SK model are studied following the same line discussed here. Moreover, since it is known that a (marginally) stable solution for the SK model requires an infinite-number of replica symmetry breakings, the requirement of local stability may partially or totally washout the results from the annealed approximation. A complete calculation of the complexity of the SK model must include full replica-symmetry breaking, making not only the calculation but also the analysis of the saddle points more difficult [7–9,21]. Supersymmetry requirements introduce partial simplifications, since they lead to a connection between the complexity and the replica calculation. However, we stress that supersymmetry is not an a priori requirement for the complexity. Indeed, while $\tilde{\rho}(f)$ is supersymmetric, the reduction procedure needed to go from $\tilde{\rho}(f)$ to $\rho(f)$ may destroy the supersymmetry, so that supersymmetry must be proved case by case.

It can be shown, see [5,6,12], that under rather broad assumptions, the supersymmetry leads to relations among the order parameters of the complexity functional reducing the number of independent parameters. For the pSP-SG model discussed here it is possible to describe the whole complexity by means of just one parameter, suggesting that the supersymmmetry is unbroken. A direct calculation [12] shows that this is indeed the case. We refer the interested reader to the cited papers for more details on this point.

Appendix A

Here we calculate $\langle \det \underline{A} \rangle$ for $N \to \infty$ using the identity (see, e.g., Ref. [27])

$$\det \underline{A} = \int \prod_{i=1}^{N} d\eta_i \, d\eta_i^+ \, \exp\left(\sum_{ij} \eta_i^+ A_{ij} \eta_j \right) \tag{46}$$

where η_i and η_i^+ are anti-commuting (Grassmann) variables. From equation (21) we have

$$\begin{aligned}
\langle \det \underline{A} \rangle = & \int \prod_{i=1}^{N} d\eta_i \, d\eta_i^+ \, \exp\left(a(q) \sum_i \eta_i^+ \eta_i \right) \\
& \times \prod_{i_1 < \cdots < i_p} \left\langle \exp\left[-\frac{\beta}{(p-2)!} J_{i_1,\ldots,i_p} \right. \right. \\
& \left. \left. \times \sum_{\pi} \eta^+_{\pi(i_1)} \eta_{\pi(i_2)} m_{\pi(i_3)} \cdots m_{\pi(i_p)} \right] \right\rangle \\
= & \int \prod_{i=1}^{N} d\eta_i \, d\eta_i^+ \, \exp\left(a(q) \sum_i \eta_i^+ \eta_i \right) \\
& \times \prod_{i_1 < \cdots < i_p} \exp\left[\frac{\mu}{2N^{p-1}} \frac{p!(p-1)!}{(p-2)!^2} \right. \\
& \times \sum_{\pi} \eta^+_{\pi(i_1)} \eta_{\pi(i_2)} m_{\pi(i_3)} \cdots m_{\pi(i_p)} \\
& \left. \times \eta^+_{i_1} \eta_{i_2} m_{i_3} \cdots m_{i_p} \right]
\end{aligned} \tag{47}$$

where $\sum_\pi$ is a sum over all permutations of p different integers $i_1, \ldots, i_p$. When the products are expanded only terms which contain pairs of $\eta_i^+ \eta_i$ with the same index survive. Since since there are $(p-2)!$ terms with the same

pairs of Grassmann variables, we end up with

$$\langle \det \underline{A} \rangle = \int \prod_{i=1}^{N} d\eta_i \, d\eta_i^+ \, \exp\left(a(q) \sum_i \eta_i^+ \eta_i \right) \times \exp\left[\frac{\mu}{2N^{p-1}} \frac{p!(p-1)!}{(p-2)!} \times \sum_{i_1<\cdots<i_p} \eta_{i_1}^+ \eta_{i_1} \eta_{i_2}^+ \eta_{i_2} m_{i_3}^2 \cdots m_{i_p}^2 \right]$$
$$= \int \prod_{i=1}^{N} d\eta_i \, d\eta_i^+ \, \exp\left[a(q) \sum_i \eta_i^+ \eta_i + \frac{\mu(p-1)}{2N} q^{p-2} \left(\sum_i \eta_i^+ \eta_i \right)^2 \right] \quad (48)$$

where we have used $Nq = \sum_i m_i^2$. The square in the exponential can be open using a Stratonovich-Hubbard transformation. The resulting expression is diagonal in $\eta_i^+ \eta_i$ and the integral over the Grassmann variable can be easily done. After a simple algebra we get

$$\langle \det \underline{A} \rangle = \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{2\pi\sigma^2(q)/N}} \times \exp N \left[-\frac{z^2}{2\sigma^2(q)} + \ln(a(q) + iz) \right] \quad (49)$$

where $\sigma^2(q) = \mu(p-1)q^{p-2}$. Finally performing the change of variable $iz + (1-q)\sigma^2(q) = ix$ we end up with [30]:

$$\langle \det \underline{A} \rangle = \frac{\exp\left(\frac{\sigma^2(q)(1-q)^2}{2} \right)}{\sqrt{2\pi\sigma^2(q)/N}} \int_{-\infty}^{+\infty} dx e^{NG(x)} \quad (50)$$

$$G(x) = -\frac{x^2}{2\sigma^2(q)} - ix(1-q) + \ln\left(\frac{1}{1-q} + ix \right). \quad (51)$$

For $N \to \infty$ the integral can be done by saddle point method:

$$\frac{dG(x)}{dx} = x \left[\frac{(1-q)^2}{1+ix(1-q)} - \frac{1}{\sigma^2(q)} \right] = 0 \quad (52)$$

which admits two solutions: $x = 0$ and $x \neq 0$. Stability requires that the saddle point be a maximum:

$$\frac{d^2G(x)}{dx^2} = \frac{(1-q)^2}{[1+ix(1-q)]^2} - \frac{1}{\sigma^2(q)} < 0. \quad (53)$$

For the $x = 0$ solution this implies that (cf. Eq. (18))

$$x_{\rm P} = 1 - \sigma^2(q)(1-q)^2 > 0 \quad (54)$$

and $G(x)$ reduces to (28).

It is easy to see that the $x \neq 0$ solution is stable only if

$$x_{\rm P} = 1 - \sigma^2(q)(1-q)^2 < 0 \quad (55)$$

in which case $G(x)$ reduces to (29).

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22. Not to be confused with the angular part of m_i used in Section 3. We leave it like this because it is the standard notation in literature
23. Usually this function is written as $G_{x_{\rm P}} = -B(1-q) + B^2/\lambda + \ln[1/(1-q)+B]$. The two expressions correspond to $B = 0$ and $B \neq 0$, respectively
24. All integrals are well defined and no extra conditions must be added
25. The coefficient in (30) contains the term $|\partial_q f(q)|_{q=q^*}|$, where $f(q)$ is the value of the TAP functional at its stationary point and not the TAP functional itself. A simple calculation shows that $\partial_q f(q) = -(q-q_{\rm a})\, x_{\rm P}(q)/2\beta(1-q)^2$ which is vanishes only for $q = q_{\rm a}$ or $x_{\rm P} = 0$
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30. By setting $ix = -B$ one recovers the standard expression in terms of the variable B. We prefer to use x since in this way the integral is over the real axis